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## Interdisciplinary Teaching of Mathematics with Primary Historical Sources

*Richard A. Edwards*

### A Great Divide

In 1959, the chemist and novelist C. P. Snow (1905–1980) identified what he saw as an increasing and unproductive isolation between scholars of different disciplines. “We have two polar groups: at one pole we have the literary intellectuals, at the other scientists. Between the two there is a gulf of mutual incomprehension.” [Sn, p. 4] Snow recalled moments in his career when literary elites would scoff at scientists who were unfamiliar with the sonnets of Shakespeare, while they were themselves ignorant of comparable scientific ideas such as the laws of thermodynamics.

Whether or not Snow accurately described intellectual life in the mid-20th century, and there is an argument that the way he concretized the gulf merely increased academic tribalism, I am fortunate to work at an institution that actively encourages interdisciplinary research and teaching. I have derived great benefit from rubbing shoulders with colleagues in the humanities and social sciences. Their perspectives on education, history, philosophy, and ethics have shaped my views on what effective teaching looks like, and what it means to learn.

What of my students? Do they appreciate interdisciplinary teaching, or wonder about how scientists and novelists can productively collaborate? Many of my students take a consumerist view toward their courses. By this I

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mean that their primary goal is to pass my class and get on with their degree. The actual content of the course is less important than the fact that it moves them one step closer to their career aspirations. If they think about it at all, many tend to think of mathematics primarily as a tool for solving problems in science. They might enjoy my class, but very few of them will think much more about it once the semester has come to a close. Is there a place in my classroom for the humanities? When I began my career this wasn't something I thought about. The only teaching I did that could be considered interdisciplinary amounted to little more than occasionally teaching a history of mathematics course, and infusing my calculus lectures with anecdotes (of questionable veracity) about famous mathematicians. Then a quote from a British educator named Charlotte Mason (1842–1923) captured my imagination:

There is a region of apparent sterility in our intellectual life. Science says of literature, *I'll have none of it*, and science is the preoccupation of our age. When we present theorems divested to the bone of all superfluous trappings, we lose the vitality along with what we've stripped away. History expires in the process, poetry cannot come to birth, religion faints; we sit down to the dry bones of science and say, *here is knowledge, all the knowledge there is to know*. [Ma, p. 317]

Snow challenged his associates in the humanities, while Mason chided the scientists. Can I, as a mathematician, teach in ways that draw from the best of both traditions? I'd like to teach in ways that retain the vitality and effervescence of mathematics. I would like to restore humanity to theorems. I want my students, as Polya described, to experience the tension and enjoy the triumph of mathematical discovery [Po].

I believe I have found a path forward by teaching mathematics via primary historical sources. Moving toward teaching in this way has been one of the most rewarding, yet challenging, efforts of my career. It has also changed how I personally think about learning mathematics.

### Primary Source Projects

The benefits and challenges of teaching with primary sources have long been a source of discussion among those interested in the history and pedagogy of mathematics [Ja]. Reading primary source texts allows students to see how individuals first conceptualized an idea, and how mathematical ideas have evolved over time. Many textbooks, almost by their very nature, present mathematical ideas as refined and finished products. In contrast, original sources help to foreground the motivations, cultural contexts, and intellectual atmosphere of their source authors. Primary sources display the patterns of communication that have

characterized the mathematical community, can reveal how those standards have changed over time, and why.

In response to these benefits, over 100 primary source projects (PSPs) have been developed under the NSF-funded TRIUMPHS project (<https://digitalcommons.ursinus.edu/triumphs/>) and its predecessor grants. PSPs are classroom projects designed to replace standard textbook-driven presentations of important mathematical topics. Each PSP features selections from one or more historical sources, supplementary text from the project author that provides both historical and mathematical background, and a series of tasks which help students interact productively with the historical source and learn its mathematical content. Each PSP has been designed to help students reach a level of fluency with a mathematical topic that is at least as strong as if they had learned it via a traditional textbook approach, in roughly the same amount of time. They are replacements, not additions, to my syllabus.

The tasks in a PSP offer students opportunities to engage with mathematics in a variety of powerful ways. These include activities which model how mathematicians actually work; for example, conjecturing, testing, refining, proving, and generalizing relationships between objects. PSPs also include tasks that allow students to interpret results as they were originally presented, and then reformulate these results in modern terms. Such activities encourage robust understanding of mathematics by immersing students in an ongoing conversation which can sometimes span centuries.

For example, I implement “Fermat’s Method of Finding Maxima and Minima” [Mo] in order to help students better understand the extreme value theorem, learn methods for finding extrema of functions, and practice their derivative rules. In addition to these object-level themes, the project can help break students out of recipe-thinking with regards to optimization, show how a technique has evolved over time, and generate discussion around the question of what counts as a general method. The source material comes from the writings of Pierre de Fermat (1607–1665), along with some commentary on Fermat’s work that Rene Descartes (1596–1650) sent to Marin Mersenne (1588–1648). Students get a glimpse into the personalities of these mathematicians as they struggle to understand and explain a topic (optimization) that first-year college students struggle to understand and explain today. After presenting his optimization process, Fermat boasted:

We can hardly be provided with a more general method.

However, at this early stage in the project, most of my students appreciate the critique raised by Descartes (in a private letter to Mersenne):

If he [Fermat] speaks of wanting to send you still more papers, I beg of you to ask him to think them out more carefully than those preceding!

I like to use these primary source excerpts to motivate student discussion: Was Fermat’s method robust, or did it only work for the specific examples he chose? Was Descartes right to question the generalizability of the method? How is Fermat’s method similar to, or different from, the method in our modern textbook? By the end of their correspondence on this subject, Descartes seemed happy with Fermat’s method, and wrote,

Seeing the last method that you [Fermat] use for finding tangents to curved lines, I can reply to it in no other way than to say that it is very good and that, if you had explained it in this manner at the outset, I would not have contradicted at all.

Certainly a ringing endorsement for students and faculty alike to communicate our ideas clearly...and show our work.

Primary source projects take students to pivotal moments in the history of mathematics. For example, the PSP “Rigorous Debates Over Debatable Rigor: Monster Functions in Introductory Analysis” [Ba] transports students to the late nineteenth century when Jean Gaston Darboux (1842–1917) and Guillaume-Jules Hoüel (1823–1886) were beginning to think about *properties of functions* as something worthy of study in their own right. As with every PSP in the TRIUMPHS collection, the goal is to teach mathematics, not its history. This PSP features core object-level themes such as continuity, differentiability, the Intermediate Value Property, Darboux’s theorem, and uniform differentiability.<sup>1</sup> Because the results are presented in their human and historical contexts, instructors can use it to talk about important meta-level themes such as: Why might someone take a critical view of the basic ideas of calculus? Why did mathematicians need to develop new vocabulary, techniques, and theorems in calculus? Other scholars, notably [BCC], have analyzed this particular project in detail with respect to its ability to promote student discussion of metadiscursive rules in Introductory Analysis. Here I restrict myself to sharing some excerpts which illustrate how the project presents the human drama of mathematical correspondence.

The discussion between Darboux and Hoüel began cordially enough...but then descended into a flurry of colorful phrases as the mathematicians become increasingly frustrated with each other.

Darboux began:

Go on then and explain to me a little, I beg you, why it is that when one uses the rule for

<sup>1</sup>In many modern texts the notion of uniform differentiability does not appear explicitly. A function is continuously differentiable if and only if its derivative is uniformly continuous.

composition of functions, the derivative of  $y = x^2 \sin \frac{1}{x}$  is found to be  $-\cos \frac{1}{x} + 2x \sin \frac{1}{x}$ , which is indeterminate for  $x = 0$  even though the true value is  $\lim_{x \rightarrow 0} \frac{y}{x} = 0$ ?

Darboux to Hoüel, January, 1875

Darboux then critiqued certain proofs which Hoüel had previously provided, using delightful phrases such as “Here is what I reproach in your reasoning. . .” and “. . . your proof has nothing but the appearance of rigor.” Hoüel’s responses hinted at a growing frustration over what he perceived as Darboux throwing up pointless counterexamples, and it seems as if the two correspondents were beginning to talk past each other.

Yes, I admit as a fact of experience (without looking to prove it in general, which might be difficult) that in the functions that I treat, one can always find  $h$  satisfying the inequality  $\frac{f(x+h)-f(x)}{h} - f'(x) < \epsilon$ , no matter what the value of  $x$ , and I avow to you that I am ignorant of what the word derivative would mean if it is not this. I believe this hypothesis is identical with that of the existence of a derivative.

Hoüel to Darboux, January 1875

Hoüel’s response did not satisfy Darboux, who seemed more concerned with attending to the dependencies between variables in a proof. He tried to get Hoüel to reflect on how variables are introduced, especially those variables that carry universal quantifiers. This is something many students today also struggle with at this point in their mathematical studies, which is one reason why having them read this source material can be powerful.

You have not addressed the nature of my objection. . . For your methods to be sound, you will need to explain very clearly what part of your reasoning is deficient in this particular case. Without that, your proofs are not proof. As for the question of the derivative, this time you change the question. It is clear that for a value  $x_0$  of  $x$ , that saying

$$\lim \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$$

is the same as saying: One can find  $h$  such that

$$\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) < \epsilon$$

for this value of  $h$  and for all values that are smaller. But there is an abyss between this proposition and the following: Being given a function  $f(x)$  for which the derivative exists for all values of  $x$  between  $a$  and  $b$ , to every quantity  $\epsilon$ , one can find a

corresponding quantity  $h$  such that

$$\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) < \epsilon$$

for all values of  $x$  between  $a$  and  $b$ .

Darboux to Hoüel, January, 1875

In addition to exploring issues such as the proper placement of quantifiers in assertions involving multiple variables (e.g., those that define today’s properties of pointwise and uniform differentiability), [BCC] note that this PSP gives students opportunities to discuss important questions related to mathematics: What is the purpose of examples? What intuitions are refined by studying them? The Darboux-Hoüel correspondence represents an important turning point in the history of analysis. I love using PSPs such as this one, which take students to the forefront of mathematical developments. I’m convinced that living on this ragged edge is both more exciting for students, and more mathematically satisfying, than some textbook-driven lectures.

Although many PSPs feature work from names my students recognize, such as Euler, Gauss, Cauchy, etc., many of the primary source texts give students exposure to geographically and culturally diverse authors. When I implement *A Genetic Context for Understanding the Trigonometric Functions* [Ot], my precalculus students get to work through selections from Greek mathematicians Hipparchus and Ptolemy, from Hindu mathematicians such as Varāhamahira, and read selections from *The Exhaustive Treatise on Shadows*, written in the court of a Turkish sultan in the year 1021. My second-semester calculus students solidify their understanding of series convergence through the PSP *Bhāskara’s Approximation to and Mādhava’s Series for Sine* [Mo2]. I’ll never forget the day when six of my students simultaneously burst out in delightful surprise at seeing source material written in Sanskrit—a language they had learned to read growing up in India (of course the PSP also provides an English translation). In my multivariable calculus course, we often end the semester with *Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green’s Theorem* [Ed] in which students read the work of the enigmatic George Green (1793–1841), a working-class miller who didn’t begin his college career until middle age, but whose ideas about electromagnetism led to the theorem which now bears his name. This year, I look forward to implementing at least one project based on the work of Maria Agnesi [Mo3].

### Challenges, and What it Means to “Learn”

Despite their great potential, teaching with PSPs brings its own set of challenges. Many of my students struggle with reading. Asking them to read (translated) excerpts from long ago is a heavy lift for some of them. It helps to have them work through the projects in groups. It takes me

longer to prepare for class when I'm going to teach with a PSP, although each project features detailed notes for instructors as well as an implementation plan. I don't use a PSP for every topic in a course (although I know instructors who teach courses using only PSPs). In a typical semester, I find time for three or four projects, depending on the course. One method I've recently found success with is putting students into groups and giving each group a different PSP (but all related to a similar topic, such as series convergence) to complete. At the end of the week, we have a "PSP Showcase" in which each group gets to do a mini-presentation of their work to the rest of the class.

Not all of the PSPs are well-aligned with my online homework problem system. However, since each PSP is intended to be a replacement for my standard lesson, I can also use the student's written work to replace the online homework for that lesson. One challenge that I am very aware of is trying to avoid interpreting history through the prejudices of today. This kind of presentism—judging the past by today's standards—can inadvertently give students a sense that all of history was an inevitable sequence of events, of which the 21st century student is its pinnacle.

The greatest—but most rewarding—challenge that I have had to wrestle with in teaching with PSPs has been re-orienting my thoughts about what it looks like to *learn* mathematics. Of course I want my students to become fluent in the discourse of modern mathematics. Yet being familiar with the modern conception of an idea can sometimes be tantamount to knowing only the last page of a long and richly complex story. Instead, I find it helpful to conceptualize learning as increased participation in the mathematical community [La]. This includes knowing both the current standards of our community, but also how our mathematical ideas have changed over time. PSPs are one means to facilitate that kind of learning.

PSPs give students opportunities to witness mathematicians at work, to "imitate [their] moves while trying to figure out the reasons for the strange things [they are] doing" [Sf, p. 202]. This may be an important step in helping students tell new stories about the world of mathematics, and their own place in that world. I close with a quote attributed to Descartes that frequently comes to mind while I'm teaching:

Scientific truths are battles won.

—Rene Descartes, quoted in [We, p. 162]

His words remind me that much of what we teach has a rich and important history. Perhaps by immersing students in that history, we can help give them a more robust understanding of our subject. PSPs may not bridge Snow's "gulf of mutual incomprehension," but I can testify to their ability to generate enthusiasm and excitement for learning mathematics.

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## Getting Your Hands Dirty: Teaching Math Biology with Active Learning Strategies

*Adrian Lam*

Nowadays, mathematical and computational methods are ubiquitous in many areas of biological research, such as genomics, ecology, evolutionary biology, neuroscience, and systems biology, to name a few. It is therefore important to introduce students to the interdisciplinary field of mathematical biology at an early stage, typically during their freshman or sophomore years. It is no surprise that an increasing number of universities are recognizing the importance of mathematical biology and integrating it into their undergraduate curriculum. The integration of mathematics and biology opens a world of possibilities for students to explore various biological phenomena using quantitative techniques. By grounding the mathematical concepts in real-life biological scenarios, students also gain a deeper appreciation for the role of mathematics in shaping their understanding of the living world.

I am an associate professor at the department of mathematics at OSU. My research interest lies in the analysis of partial differential equations. I have worked on systems of reaction-diffusion equations and free-boundary problems which are inspired by applications in biology. I have had the pleasure of teaching and advising students in the math bio track since I joined the faculty at the Ohio State University in 2014. In this article, I aim to share some of my personal experiences with teaching the course Introduction to Mathematical Biology with my colleague Avner Friedman (founding director of Mathematical Biosciences Insti-

tute) since 2018. While coteaching mathematical biology can be quite different from teaching other more traditional mathematics courses, in terms of the syllabus, audience, and teaching goals, it also offers ample opportunities to apply active learning techniques. Here, I would like to share some of our recent experiences and personal take-aways in interacting with our students.

One of the major differences in teaching mathematical biology compared to traditional mathematics courses lies in the scope and emphasis of the syllabus. In a math biology course, it is crucial to provide students with a thorough understanding of the biological context behind mathematical models. Traditional applied math courses may mention motivation briefly before diving into theorems and proofs. However, in a math biology class, the goal is to establish a strong connection between mathematical methods and their applicability to biological problems. Rigorous proofs are still valuable but take a back seat to explaining the biological rationale behind the models.

Another significant difference is the audience in a math biology course. Many students who enroll in this course do not major in mathematics. While they can be bright individuals (many of whom are premed students), they often find mathematical concepts challenging to grasp or even intimidating. As instructors, building rapport with such students and ensuring effective communication can be a challenging but rewarding task.

The traditional way of teaching mathematics involves presenting the subject logically, defining precise mathematical objects, deriving results, and providing examples of alternative solution methods. After that, students can work on problem sets independently to improve their familiarity with the techniques. For a math biology class, however, there are opportunities to apply active learning methods and dedicate more time to inquiry- and problem-based labs.

To foster a more interactive and engaging learning environment, we structured our course into weekly modules. Each week, we introduce a mathematical method alongside one or more biological motivations for its use. For instance, for the module focused on epidemiology in week 6, we introduce the SIR (Susceptible-Infected-Recovered) model, which is a set of ordinary differential equations depicting the transition of the overall infection status among members of a population. Given that most students were affected by the COVID-19 pandemic, lively discussions ensued when exploring how to incorporate real-world details into the model, such as how to incorporate an asymptomatic period in the SIR model before an infected individual becomes symptomatic and can be detected.

In addition to theoretical discussions, we emphasize numerical computation during our teaching. This approach allows students to witness how models work and how to

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